Advanced Macro I Fall 2013
Shanghai University of Finance and Economics
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Assignment 2

The due date for this assignment is Tuesday, October 22.

1. (*Two-sector growth model*) Consider the following two-sector model of optimal growth. A social planner seeks to maximize the life-time utility of the representative HH given by $\sum_{t=0}^{\infty}\beta^t u(c_t, l_t)$, where c_t is consumption of good 1 at time t, whereas l_t is leisure at time t.

   Sector 1 produces consumption good using capital k_{1t} and labor n_{1t} according to the production function $f_1(k_{1t}, n_{1t})$. Sector 2 produces capital good which is going to be used in the production next period according to the production function $f_2(k_{2t}, n_{2t})$. The time endowment for the HH at each period is $\bar{l}$. The initial capital stock is given by $k_0 > 0$.

   (a) Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

   (b) Consider another economy that is similar to the previous one except for the fact that capital is sector specific, i.e., capital stock for sector 1 must be only used for this sector, same for sector 2. The capital-good sector produces capital that is specific to each sector according to the transformation technology

   $$g(k_{1,t+1}, k_{2,t+1}) \leq f_2(k_{2t}, n_{2t}).$$

   Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

2. (*Habit Persistence*) Consider following dynamic problem with habit persistence preference:

$$\max \sum_{t=0}^{\infty} \beta^t (\ln c_t + \gamma \ln c_{t-1})$$

$$s.t.$$

$$\begin{aligned} c_t + k_{t+1} &\leq Ak_t^{\alpha} \\ c_t, k_{t+1} &\geq 0 \\ k_0, c_{-1} &> 0 \text{ given} \end{aligned}$$

   Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

3. (*Howard's policy iteration algorithm*) Consider the following optimal growth problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$s.t.$$

$$\begin{aligned} c_t + k_{t+1} &\leq Ak_t^{\alpha}\theta_t \\ k_0 \text{ given, } A &> 0, 1 > \alpha > 0 \end{aligned}$$

where the sequence $\{\theta_t\}$ is an i.i.d. shock with $\ln \theta_t$ distributed according to a normal distribution with mean zero and variance σ^2.

Consider the following algorithm. Guess at a policy of the form

$$k_{t+1} = h_0 A k_t^\alpha \theta_t$$

for any constant $h_0 \in (0, 1)$. Then form the value function

$$v_0(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^\alpha \theta_t - h_0 A k_t^\alpha \theta_t)$$

(Hint: you will see the value function takes the form $v_0(k_0, \theta_0) = H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1-\alpha\beta} \ln k_0$.) Next, we choose a new policy h_1 by maximizing

$$\ln(A k^\alpha \theta - k') + \beta E v_0(k', \theta')$$

where $k' = h_1 A k^\alpha \theta$. Then we form the value function again

$$v_1(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^\alpha \theta_t - h_1 A k_t^\alpha \theta_t).$$

Continue iterating on this scheme until successive hj have converged.

Show that, for the present example, this algorithm converges to the optimal policy function in one step.

4. (*Guess and Verify*) Consider a social planner who faces the following problem:

$$\begin{aligned}
& \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& s.t. \\
c_t + i_t &\leq F(k_t, l_t) \\
k_{t+1} &= (1-\delta) k_t + i_t \\
c_t &\geq 0, 0 < l_t \leq 1 \\
& k_0 \text{ given}
\end{aligned}$$

where utility function takes the form

$$u(c_t) = c_t - \theta c_t^2, \theta > 0$$

Assume that c is always in the range where $u'(c)$ is positive. Output is linear in capital, $F(k, l) = Ak$. Ans we assume $\delta = 0$, i.e., no depreciation at all.

(a) Wirte down the Bellman equation for this problem. Be clear what are the state variables, what arr control variables.

(b) Derive the Euler equation relating c_t and expectations of c_{t+1}.

(c) Guess the consumption takes the form $c_t = H + F k_t$. Given this guess, what is the policy function for k_{t+1}?

(d) What is the value for the parameters H and F?

5. (*Guess and Verify again*) Consider the following problem

$$\begin{aligned} & \max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ A_{t+1} & \leq R_t(A_t - c_t) \\ A_0 & > 0 \text{ given.} \end{aligned}$$

where utility function takes the form

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \alpha > 0$$

and R_t is an i.i.d. shock such that $ER_t^{1-\alpha} < \frac{1}{\beta}, 1 > \beta > 0$. It is assumed that c_t must be chosen before R_t is observed. Show that the optimal policy function takes the form $c_t = \lambda A_t$ and give an explicit formula for λ. (Hint: Guess the value function takes the form $v(A) = BA^{1-\alpha}$ for some constant B.)

6. (*Optimal growth model with two types of agent*) Consider the standard optimal growth model with production function $F(k_t, l_t)$. Suppose now that there are an equal number of two types of HHs with preferences given by $\sum_{t=0}^{\infty} u_1(c_{1t})$ and $\sum_{t=0}^{\infty} u_2(c_{2t})$ respectively. Initial ownership of the capital stock is given by k_{10} and k_{20}.

   (a) Define the social planner's problem for this economy.

   (b) Define an ADE.

   (c) Set up the social planner's problem as a dynamic programming problem.

   (d) Define a SME.

7. Consider a social planner who faces the following problem:

$$\begin{aligned} & \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ c_t + i_t & \leq F(k_t, l_t) \\ k_{t+1} & = (1-\delta)k_t + i_t \\ c_t & \geq 0, 0 < l_t \leq 1 \\ & k_0 \text{ given} \end{aligned}$$

Now we assume $u(c) = \ln c$, production function $F(k_t, l_t) = k_t^{\alpha} l_t^{1-\alpha}$ and $\delta = 1$. Establish that you indeed have a solution for this problem. (Hint: I am not asking you to solve the solution, instead I am asking you to prove the existence of a solution to the sequential problem above.)

8. (*Recursive Competitive Equilibrium with labor-leisure choice*) Consider the following optimal growth model with labor-leisure choice

$$\begin{aligned}
& \max_{\{c_t,l_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
& s.t. \\
c_t + i_t &\leq F(k_t, n_t) \\
k_{t+1} &= (1-\delta)k_t + i_t \\
c_t &\geq 0, n_t, l_t > 0, n_t + l_t = 1 \\
& k_0 \text{ given}
\end{aligned}$$

where n_t is the labor input and l_t denotes leisure.

(a) Define a SME for this economy.

(b) Define a RCE for this economy.

9. (*Value function iteration*) Consider the following optimal growth problem

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t \ln c_t \\
& s.t. \\
c_t + k_{t+1} &\leq k_t^{\alpha} \\
c_t, k_{t+1} &\geq 0, k_0 \text{ given}
\end{aligned}$$

We assume $\beta = 0.97, \alpha = 0.36, \delta = 0.06$. Write a Matlab code that takes $k_0 = 0.75 \times k^*$ and iterates towards k^*, where k^* is the steady state value of capital stock. Using $\varepsilon = 0.0001$ as a convergence criterion. Plot the value function $v(k)$ and the optimal policy function $k' = g(k)$. How many time periods does it take to converge to the steady state? Do the same thing from $k_0 = 0.10 \times k^*$ and compute the number of time periods to converge. Now change α from 0.36 to 0.30, compute the number of time periods to convergence. How does the speed of convergence depend on α? Why?

10. (*Wealth inequality in a two-period economy*) Suppose that there are two types of consumers distinguished by their initial endowments of capital. In particular, type-1 consumers (who comprise fraction θ of the population) are richer than type-2 consumers (who comprise fraction $1-\theta$ of the population): type-1 consumers are endowed with k_0^1 units of capital and type-2 consumers are endowed with k_0^2 units of capital, where $k_0^1 > k_0^2$. The two types of consumers are identical in all other respects. Each consumer takes prices as given (in particular, each consumer takes the aggregate, or total, capital stock in period 1 as given) when making savings decisions in period 0. The equilibrium (or consistency) condition is that the total savings of the two types of consumers in period 0 must equal the aggregate capital stock that consumers take as given when deciding how much to save. Assume that each consumer's utility function takes the form $u(c_0) + \beta u(c_1)$ with $u(c) = \log(c)$. The production technology available to firms is: $y = k^{\alpha} n^{1-\alpha}$, with $1 > \alpha > 0$, where y is the firm's output and k and n are the services of capital and labor, respectively.

(a) Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (i.e., the parameters α, θ and β, and the initial capital stock k_0^1 and k_0^2).

(b) Use your answer to part (a) to show that changes in k_0^1 and k_0^2 that keep aggregate capital in period 0 (i.e., $\theta k_0^1 + (1-\theta)k_0^2$) constant have no effect either on equilibrium aggregate savings or on equilibrium prices. This is a version of an *aggregation theorem* for this economy: holding the total amount of capital in period 0 constant, the behavior of the aggregates in this economy does not depend on the distribution of capital in period 0.

(c) Suppose that the felicity function takes the form: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ where $\sigma > 0$. Does an aggregation theorem like the one described in part (b) hold for this economy? Explain why or why not.